ISRAEL JOURNAL OF MATHEMATICS, Vol. 47, No. 4, 1984

# ON THE UNIFORMIZATION OF THE SOLUTIONS OF THE POROUS MEDIUM EQUATION IN $\mathbf{R}^N$ †

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ABSTRACT
The asymptotics of the solution of the porous medium equation are related to the size of the initial data measured in an optimal way. The universality of the separable solutions is established. Finally an interesting difference with the heat equation is pointed out.

In this paper we are concerned with the asymptotic behavior as $t \to \infty$ of solutions of the porous medium equation with nonnegative, nonintegrable initial data:

$$\text{(PM)} \qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta(u^m), \qquad m > 1, \\ &u(x,0) = u_0(x) \geqq 0, \qquad u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N). \end{aligned}$$

It is helpful for the understanding of the various hypotheses involved to begin by describing the corresponding problem for the heat equation

$$\text{(HE)} \qquad \begin{aligned} &\frac{\partial u}{\partial t} = \Delta u, \\ &u(x,0) = u_0(x) \geqq 0, \qquad u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N). \end{aligned}$$

It is shown in [17] that the solution $u(x,t)$ of (HE) stabilizes to some number $a$, i.e.

$$\lim_{t \to \infty} u(x,t) = a \qquad \text{for each } x \text{ in } \mathbf{R}^N$$

if and only if $u_0$ satisfies

† Supported in part by the National Science Foundation under Grant MCS-800L540.
Received May 31, 1983 and in revised form February 2, 1984

(AV) $$\lim_{r\to\infty} |B_r(0)|^{-1}\int_{|x|<r} u_0(x)dx = a.$$

In contrast, the counterexample in Section 2 of this paper shows that for solutions of the porous medium equation (PM) the hypothesis (AV) alone does not imply convergence of $u(x,t)$ to the "average" $a$ in any sense. To remedy this situation we introduce an alternative method of measurement of the average via the condition

(*) $$\lim_{\xi\to\infty} u_0(\xi x) = a, \qquad \xi \in \mathbf{R}$$

which is to be understood in the sense of distributions $\mathcal{D}'(\mathbf{R}^N)$. We then have

PROPOSITION A. *Let $u(x,t)$ be a solution of* (PM). *Then*

$$\lim_{t\to\infty} u(x,t) = a$$

*uniformly on the expanding sets $\{x \in \mathbf{R}^N : |x| < Ct^{1/2}\}$, if and only if* (*) *holds.*

This is a special case of the more general Theorem B below. Let's point out here that (*) implies, but is not implied by, (AV). See the remark following Proposition C. The function $u_0(x) = 1 + \operatorname{sgn}(x)$, $x \in \mathbf{R}$, for instance satisfies (AV) with $a = 1$, but (*) does not hold. The computation in Section 2 confirms that $u(x,t) \not\to 1$ as $t\to\infty$. On the other hand the function $u_0(x) = 1 + \sin x + \tilde{u}(x)$,

$$u(x) = \begin{cases} n & \text{when } n^3 - 1 < x < n^3 \quad (n = 1,2,3,\dots) \\ 0 & \text{otherwise} \end{cases}$$

oscillates, grows at infinity, and satisfies (*) with $a = 1$, hence Proposition A applies.

The generalization of Proposition A that we have in mind corresponds to modifying (AV) so that it reads

(GAV) $$\lim_{r\to\infty} |B_r(0)|^{-s/N}\int_{|x|<r} u_0(x)dx = l$$

for some $s \geqq 0$. The exponent $s$ is a measure of the rate of growth or decay of $u_0(x)$ as $|x|\to\infty$. In this context an important role is played by the family of functions $\Lambda_s$, $s > 0$ defined by

$$\Lambda_s(x) = \frac{s}{N|B_1(0)|^{1-s/N}}|x|^{s-N}, \qquad x \in \mathbf{R}^N - \{0\}.$$

Passing to the limit at $s=0$ we also define $\Lambda_0(x)=\delta(x)=$ Dirac's delta function. By the existence theorem of [5] for each $s\in[0,2/(m-1)+N)$ the porous medium equation has a unique solution $Q_s(x,t)$ defined for all $t>0$ corresponding to the initial data $\Lambda_s(x)$. Thus $q(x,t)\equiv lQ_s(x,l^{m-1}t)$ solves (PM) with the initial data $l\Lambda_s(x)$ for any $l>0$. The functions $q(x,t)$ have a universal nature in the sense that they approximate solutions of (PM) for a large class of initial data. Specifically, suppose that $u_0\in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $u_0\geqq 0$, and that there exists an $s\in[0,2/(m-1)+N)$ and an $l\geqq 0$ such that

$$\text{(H)}\qquad \xi^{N-s}u_0(\xi x)\to l\Lambda_s(x)\qquad \text{as } \xi\to\infty$$

in the sense of distributions $\mathcal{D}'(\mathbf{R}^N)$. Then we have:

THEOREM B. *Let $u(x,t)$ be a solution of* (PM). *Then*

$$\lim_{t\to\infty} t^{\alpha}\,|\,u(x,t)-q(x,t)\,|=0$$

*uniformly on the expanding sets $\{x\in\mathbf{R}^n : |\,x\,|<Ct^{\beta}\}$, if and only if* (H) *holds.*

*Here $C$ is an arbitrary positive constant and*

$$\alpha=\frac{N-s}{2+(m-1)(N-s)},\qquad \beta=\frac{1}{2+(m-1)(N-s)}.$$

Observe that if $u_0$ is such that $s=N$, then (H) reduces to (*) with $l=a$. In this case $q(x,t)\equiv a$, $\alpha=0$, $\beta=\frac{1}{2}$, hence Proposition A is obtained. On the other hand, if $u_0\in L^1(\mathbf{R}^N)$, then (H) holds with $s=0$ and $l=\int_{\mathbf{R}^N}u_0(x)dx$. In this case $q(x,t)$ is the Barenblatt solution corresponding to the initial data $l\delta(x)$, thus we capture in particular the result of Friedman and Kamin [14] and Kamin [9].

The meaning of condition (H) is brought out in the following proposition.

PROPOSITION C. *A function $u_0\in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $u_0\geqq 0$, satisfies* (H) *if and only if*

$$(**)\qquad \lim_{\xi\to\infty}|\,\xi\Omega\,|^{-s/N}\int_{\xi\Omega}u_0(x)dx=l\,|\,\Omega\,|^{-s/N}\int_{\Omega}\Lambda_s(x)dx$$

*for all measurable sets $\Omega\subset\mathbf{R}^N$. Here $\xi\Omega$ denotes the dilation of $\Omega$ by a factor of $\xi$, with respect to the origin; $\xi\Omega=\{\xi x : x\in\Omega\}$.*

REMARKS. (i) In the special case $s=N$, (**) reads

$$\lim_{\xi\to\infty}|\,\xi\Omega\,|^{-1}\int_{\xi\Omega}u_0(x)dx=l$$

for all $\Omega\subset\mathbf{R}^N$, whence it follows that (*) implies (AV).

(ii) Taking $\Omega = B_1(0)$ in (**) we see that (H) implies (GAV).

(iii) When $u_0$ is spherically symmetric (radial) then it suffices to test (**) with balls centered at the origin:

PROPOSITION D. *If $u_0$ is radial then* (GAV) *and* (H) *are equivalent.*

Then significance of the conditions (H) and (GAV) stems from the fact that they describe conserved quantities under (PM):

PROPOSITION E. (i) *If $u_0$ satisfies* (H) *then the corresponding solution of* (PM) *satisfies for each $t > 0$:*

$$\xi^{N-S} u(\xi x, t) \to l \Lambda_s(x) \qquad \text{as } \xi \to \infty$$

*in $\mathcal{D}'(\mathbf{R}^N)$.*

(ii) *If $u_0$ satisfies* (GAV), *then the corresponding solution of* (PM) *satisfies for each $t > 0$:*

$$|B_r(0)|^{s/N} \int_{|x|<r} u(x,t)dx \to l$$

*as $r \to \infty$.*

Proposition E and Theorem B are proved in Section 1. Propositions C and D are proved in the Appendix and Proposition A is a special case of Theorem B.

The basic tools in our proof of Theorem B are the estimates obtained by Benilan, Crandall and Pierre [5], as well as the existence and uniqueness statements in that work.

In case $s = 0$ the initial data for $q(x, t)$ is a distribution so for uniqueness we refer to Pierre [16]. The lower bound estimate in Lemma 1.5 uses in an essential way the Harnack type inequality of Aronson and Caffarelli [3]. An alternative proof, via comparison, is given by Alikakos and Rostamian [2]. The computations for the counterexample in Section 2 depend on the work of Gilding and Peletier [11] [12]. Part of [12] is also concerned with a detailed study of functions $Q_s(x, t)$ defined above.

The fact that the dilations in conditions (H) and (**) are performed with respect to the origin is not of particular significance. It can be easily verified (using the non-negativeness of $u_0$) that conditions (H) and (**) are invariant under translation of coordinates. But of course the convergence in (H) or (**) may not be uniform with respect to the location of the origin. It can be shown that if we require (H) or (**) to hold uniformly with respect to translations of the coordinates in $\mathbf{R}^N$, then either $l = 0$ or $s = N$. In the latter case (H) reads:

$$u_0(\xi(x - x_0) + x_0) \to l \qquad \text{as } \xi \to \infty \tag{5}$$

in $\mathcal{D}'(\mathbf{R}^N)$, uniformly in $x_0 \in \mathbf{R}^N$. It may further be verified that this condition is equivalent to

$$|B_R(x_0)|^{-1} \int_{B_r(x_0)} u_0(x)dx \to l \qquad \text{as } R \to \infty \tag{6}$$

uniformly in $x_0 \in \mathbf{R}^N$. If $l > 0$ then it follows easily from Lemma 1.5 that under these uniform assumptions on $u_0$, there exist numbers $\bar{t}$ and $c > 0$ such that $u(x,t) > c$ for all $t > \bar{t}$ and $x \in \mathbf{R}^N$. Thus in particular $u(x,t)$ becomes a classical solution after a finite time. For instance, the solution corresponding to $u_0(x) = 1 + \sin x$, $x \in \mathbf{R}$, converges uniformly (in $\mathbf{R}$) to 1, while the convergence to 1 of the solution corresponding to the initial data $u_0(x) = 1 + \sin x + \tilde{u}_0(x)$ defined above is not uniform.

Asymptotic results of the type stated in Theorem B, the so-called "intermediate asymptotics", occur frequently in various contexts. We have already mentioned the works of Kamin [14] and Friedman and Kamin [5]. See also Vazquez [18] for a refinement. Similar estimates are obtained by Kamin and Rosenau [15] for the equation of propagation of thermal waves, and Gmira [13] for a nonlinear heat equation. For an analogous result when the domain is bounded, see [4].

## §1. Proofs of Theorem B and Proposition E

We begin by recalling some facts from [5]. Define $u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $r \geqq 0$, $\mu \geqq 0$:

$$\| u_0 \|_{r,\mu} = \sup_{\rho \geqq r} \rho^{-\mu(2/(m-1)+N)} \int_{|x|<\rho} |u_0(x)|\, dx.$$

Note that $\| u_0 \|_{r,\mu}$ is decreasing as $r$ increases, so let

$$l(u_0, \mu) = \lim_{r \to \infty} \| u_0 \|_{r,\mu}. \tag{1.1}$$

It is established in [5] that when $u_0 \geqq 0$, then $l(u_0, 1) = 0$ if and only if (PM) has a global in time solution. Now let $r > 0$ and $\mu \in [0,1]$ and observe

$$\begin{aligned} \| u_0 \|_{r,1} &= \sup_{\rho \geqq r} \rho^{-(1-\mu)(2/(m-1)+N)} \rho^{-\mu(2/(m-1)+N)} \int_{|x|<\rho} |u_0(x)|\, dx \\ &\leqq r^{-(1-\mu)(2/(m-1)+N)} \sup_{\rho \geqq r} \rho^{-\mu(2/(m-1)+N)} \int_{|x|<\rho} |u_0(x)|\, dx, \end{aligned}$$

therefore

$$\| u_0 \|_{r,1} \leqq r^{-(1-\mu)(2/(m-1)+N)} \| u_0 \|_{r,\mu}. \tag{1.2}$$

In particular if $u_0$ is such that $\| u_0 \|_{1,\mu} < \infty$ for some $\mu \in [0,1)$, then since $\| u_0 \|_{r,\mu} \leqq \| u_0 \|_{1,\mu}$ for $r > 1$, we get from (1.2) that $\| u_0 \|_{r,1} \to 0$ as $r \to \infty$, hence $l(u_0, 1) = 0$ thus the solution of (PM) is defined for all $t > 0$.

It is easy to see that (H) implies $\| u_0 \|_{1,\mu} < \infty$, thus the global time existence for (PM). For this, suppose (H) holds with some $s$, suppose $s \in [0, 2/(m-1) + N)$, and let

$$\mu = s \Big/ \left( \frac{2}{m-1} + N \right). \tag{1.3}$$

Then

$$\begin{aligned} \| u_0 \|_{1,\mu} &= \sup_{\rho \geqq 1} \rho^{-s} \int_{|x|<\rho} u_0(x) dx \\ &= \sup_{\rho \geqq 1} \rho^{N-s} \int_{|x|<1} u_0(\rho x) dx. \end{aligned}$$

But as $\rho \to \infty$, we have

$$\int_{|x|<1} \rho^{N-s} u_0(\rho x) dx \to l \int_{|x|<1} \Lambda_s(x) dx < \infty,$$

hence $\| u_0 \|_{1,\mu} < \infty$ as asserted.

The following lemma establishes an upper bound for solutions of (PM) satisfying $\| u_0 \|_{1,\mu} < \infty$.

Lemma 1.1. *Suppose* $\| u_0 \|_{1,\mu} < \infty$ *for some* $\mu \in [0,1)$. *Let* $u(x,t)$ *be the solution of* (PM), *and define* $s$ *as in* (1.3). *If* $l(u_0, \mu) > 0$, *then for any* $M > 1$

$$\| u(\cdot, t) \|_{L^\infty(|x| < k_1 M t^\beta)} \leqq k_2 M^{2\mu/(m-1)} t^{-\alpha}. \tag{1.4}$$

*If* $l(u_0, \mu) = 0$, *then*

$$\| u(\cdot, t) \|_{L^\infty(|x| < k_3 t^\beta)} = o(t^{-\alpha}). \tag{1.5}$$

*Here we have used the following notation:*

$$\alpha = \frac{(m-1)N - \mu[(m-1)N+2]}{(1-\mu)(m-1)[(m-1)N+2]} = \frac{N-s}{2+(m-1)(N-s)},$$

$$\beta = \frac{1}{(1-\mu)[(m-1)N+2]} = \frac{1}{2+(m-1)(N-s)},$$

$$\lambda = N/[(m-1)N+2].$$

*The positive constants* $k_1$, $k_2$ *and* $t_0$ *depend on* $m$, $N$, $\mu$, *and* $\| u_0 \|_{1,\mu}$ *while* $k_3$ *is arbitrary.*

PROOF. In [4] the following estimate is derived. If $\| u_0 \|_{1,1} < \infty$, then for all $r \geqq 1$

$$\| u(\cdot, t) \|_{L^\infty(|x|<r)} \leqq c_2 r^{2/(m-1)} t^{-\lambda} \| u_0 \|_{r,1}^{2\lambda/N} \tag{1.6}$$

provided that $t < c_1 \| u_0 \|_{r,1}^{1-m}$. Here $c_1$ and $c_2$ depend only on $m$ and $N$. Now on the one hand, we substitute in (1.6) for $\| u_0 \|_{r,1}$ in terms of $\| u_0 \|_{r,\mu}$ using (1.2), on the other hand we notice that by (1.2), (1.6) will hold in particular if

$$t < c_1 [ r^{(1-\mu)(2/(m-1)+N)} \| u_0 \|_{r,\mu} ]^{1-m}.$$

Thus fix $M > 1$ arbitrarily, and choose $r$ and $t$ such that

$$t = c_1 M^{1/\beta} r^{(1-\mu)[(m-1)N+2]} \| u_0 \|_{1,\mu}^{1-m},$$

which we rewrite as $r = k_1 M t^\beta$. The condition $r \geqq 1$ implies that $t \geqq t_0$ for some positive constant $t_0$. With this choice (1.6) becomes

$$\begin{aligned} \| u(\cdot, t) \|_{L^\infty(|x|<k_1 M t^\beta)} &\leqq c_2 (k_1 M t^\beta)^{2/(m-1)} t^{-\lambda} [ (k_1 M t)^{-(1-\mu)(2/(m-1)+N)} \| u_0 \|_{k_1 M t^\beta, \mu} ]^{2\lambda/N} \\ &= k_2 M^{2\mu/(m-1)} t^{-\alpha} \| u_0 \|_{k_1 M t^\beta, \mu}^{2\lambda/N}. \end{aligned} \tag{1.7}$$

Thus if $l(u_0, \mu) > 0$, we can replace the last term in (1.7) by $\| u_0 \|_{1,\mu}$ to obtain (1.4). If $l(u_0, \mu) = 0$, then the last term in (1.7) tends to zero as $t \to \infty$, so we obtain (1.5).

REMARK 1.2. Estimate (1.5) establishes Theorem B for $l = 0$ under the weaker hypothesis (GAV).

LEMMA 1.3. (= Proposition E(i)) *The condition* (H) *remains invariant for positive time.*

PROOF. Let $\varphi$ be a $C^\infty$ function with support in some ball $|x| < R$. Assuming for the moment sufficient smoothness on $u(x, t)$ we have for $0 < t_1 < t_2$

$$\begin{aligned} I_\xi &\equiv \int_{|x|<R} \xi^{N-s} [u(\xi x, t_2) - u(\xi x, t_1)] \varphi(x) dx \\ &= \xi^{-s} \int_{|x|<\xi R} [u(x, t_2) - u(x, t_1)] \varphi\left(\frac{x}{\xi}\right) dx \\ &= \xi^{-s} \int_{|x|<\xi R} \int_{t_1}^{t_2} u_t(x, \tau) \varphi\left(\frac{x}{\xi}\right) d\tau dx \\ &= \xi^{-s} \int_{|x|<\xi R} \int_{t_1}^{t_2} \Delta u^m(x, \tau) \varphi\left(\frac{x}{\xi}\right) d\tau dx \\ &= \xi^{-s} \int_{|x|<\xi R} \int_{t_1}^{t_2} u^m(x, \tau) \Delta\varphi\left(\frac{x}{\xi}\right) d\tau dx. \end{aligned}$$

Therefore

$$| I_\xi | \leqq c\xi^{-s-2} \int_{t_1}^{t_2} \left[ \| u(\cdot, t) \|^{m-1}_{L^\infty(|x|<\xi R)} \int_{|x|<\xi R} u(x,\tau)dxd\tau \right].$$

Now we apply the estimate (1.67) of [4] for $\mu' > \mu = s/[2/(m-1)+N]$ chosen such that $t_2 < T_{r,\mu'}(u_0)$ and $\xi R \geqq r$. We obtain

$$| I_\xi | \leqq c\xi^{-s-2+2\mu'} \int_{t_1}^{t_2} \int_{|x|<\xi R} \tau^{-\lambda(m-1)} u(x,\tau)dxd\tau.$$

On the other hand, by the analogue of the estimate (1.8) of [5] for $\mu'$, we have

$$\xi^{-\mu'[2/(m-1)+N]} \int_{|x|<\xi R} u(x,\tau)dx \leqq c \| u_0 \|_{r,\mu'}.$$

Consequently

$$| I_\xi | \leqq c\xi^{-s-2+2\mu'+\mu'[2/(m-1)+N]} \int_{t_1}^{t_2} \tau^{-\lambda(m-1)} d\tau.$$

Note that the singularity is integrable and that by taking $\mu'$ sufficiently close to $\mu$ we can make the exponent of $\xi$ negative. Taking first $\xi \to \infty$ then $t_1 \to 0$ we arrive at

$$\lim_{\xi\to\infty} \left| \int_{|x|<R} \xi^{N-s} [u_0(\xi x) - u(\xi x, t)] \varphi(x)dx \right| = 0.$$

To get rid of the extra regularity assumed of $u(x,t)$ we may replace $u_0(x)$ by $u_0(x) + \varepsilon |x|^{s-N}$, repeat the argument, and then let $\varepsilon \to 0$. Here, and implicitly everywhere we make use of continuity properties of the semigroup of solutions established in [5].

LEMMA 1.4. (= Proposition E (ii)) *The condition* (GAV) *remains invariant for positive time.*

PROOF. It is sufficient to show that

$$\lim_{R\to\infty} \int_{|x|<R} [u(x,t) - u_0(x)]dx = 0.$$

Suppose that $u_0$ satisfies (GAV). Then for any $\varepsilon > 0$ there exist numbers $R_1(\varepsilon)$ and $\delta_1(\varepsilon)$ such that

$$R^{-s} \int_{R<|x|<(1+\delta)R} u_0(x)dx < \varepsilon$$

for all $R > R_1(\varepsilon)$ and $0 < \delta < \delta_1(\varepsilon)$.

To see this, observe that

$$R^{-s}\int_{R<|x|<(1+\delta)R} u_0(x)dx = (1+\delta)^s[(1+\delta)R]^{-s}\int_{|x|<(1+\delta)R} u_0(x)dx$$
$$-R^{-s}\int_{|x|<R} u_0(x)dx$$
$$+[(1+\delta)^s-1]R^{-s}\int_{|x|<R} u_0(x)dx.$$

The first term on the right side can be made small by choosing $R$ sufficiently large (by (GAV)) and the second term can be made small by choosing $\delta$ sufficiently small, also using (GAV).

Now take a family $\{\varphi_\delta\,;\delta>0\}$ of functions in $C^\infty([0,\infty))$ such that $0\leqq\varphi_\delta(z)\leqq 1$ for $z\geqq 0$, $\varphi_\delta(z)=1$ for $z\in[0,1]$, $\varphi_\delta(z)=0$ for $z\geqq 1+\delta$. Now fix an $\varepsilon>0$ and choose and fix $\delta\in(0,\delta_1(\varepsilon))$. Proceeding as in the argument of the previous lemma, minus the hypothesis (H), we obtain

$$\int_{|x|<1+\delta}\xi^{s-N}[u_0(\xi x)-u(\xi x,t)]\varphi_s(x)dx\to 0$$

as $\xi\to\infty$. Equivalently

$$R^{-s}\int_{|x|<(1+\delta)R}[u_0(x)-u(x,t)]\varphi_\delta\left(\frac{x}{R}\right)dx\to 0$$

as $R\to\infty$. Thus pick $R_2(\varepsilon)$ such that

$$\left|R^{-s}\int_{|x|<(1+\delta)R}[u_0(x)-u(x,t)]\varphi_\delta\left(\frac{x}{R}\right)dx\right|<\varepsilon$$

for all $R>R_2(\varepsilon)$. In particular if $R>\max\{R_1(\varepsilon),R_2(\varepsilon)\}$ then

$$R^{-s}\int_{R<|x|<(1+\delta)R}u(x,t)dx\leqq\varepsilon+R^{-s}\int_{R<|x|<(1+\delta)R}u_0(x)dx\leqq 2\varepsilon.$$

Eventually

$$\left|R^{-s}\int_{|x|<R}[u_0(x)-u(x,t)]dx\right|\leqq\left|R^{-s}\int_{|x|<(1+\delta)R}[u_0(x)-u(x,t)]\varphi_\delta\left(\frac{x}{R}\right)dx\right|$$
$$+\left|R^{-s}\int_{R<|x|<(1+\delta)R}[u_0(x)-u(x,t)]\varphi_\delta\left(\frac{x}{R}\right)dx\right|$$
$$\leqq\varepsilon+4\varepsilon=5\varepsilon,$$

as asserted.

The next lemma establishes a lower bound for the solutions of (PM). Combined with Lemma 1.1 it shows that $u(x,t)$ behaves like $t^{-\alpha}$ as $t \to \infty$ when $l > 0$.

LEMMA 1.5. *Under* (GAV) (*and hence under* (H)) *and* $l > 0$ *we have the estimate*

$$u(x,t) \geqq k_5 l^{2\beta} t^{-\alpha}$$

*on the ball* $|x| < k_4 l^{(m-1)\beta} t^{\beta}$ *and all* $t > T$, *where* $\alpha$ *and* $\beta$ *are defined in Lemma* 1.1, $k_4$ *and* $k_5$ *depend only on* $m$, $N$ *and* $s$, *while* $T$ *depends on* $m$ *and* $u_0$.

PROOF. By a result of P. Sacks recorded in [5] $u(x,t)$ is continuous in $\mathbf{R}^N \times (0,\infty)$. In view of Lemma 1.3 then we may assume that $u_0$ is continuous. According to the Harnack-type estimate in [3] we have

$$\int_{|z-x|<R} u_0(z)dz \leqq c[R^{N+2/(m-1)} + u(x,1)^{1+(m-1)N/2}]$$

for all solutions of (PM), where $R > 0$ is arbitrary and $c$ depends only on $m$ and $N$. Let $u(x,t)$ be a solution of (PM). Then $w_k(x,t) = k^\alpha u(k^\beta x, kt)$ also solves (PM) for any $k > 0$. The important property of this similarity transformation is that it leaves the condition (GAV) invariant. Now fix an $x \in \mathbf{R}^N$, and an $R > 0$ such that $|x| < R/2$, observe that the ball $|z| < R/2$ is contained in the ball $|z - x| < R$ and apply the Harnack inequality to $w_k$ to obtain

$$\int_{|z|<R/2} w_k(z,0)dz \leqq c(R^{N+2/(m-1)} + w_k(x,1)^{1+(m-1)N/2}],$$

or in terms of $u(x,t)$:

$$\int_{|z|<R/2} k^\alpha u_0(k^\beta z)dz \leqq c[R^{N+2/(m-1)} + [k^\alpha u(k^\beta x, k)]^{1+(m-1)N/2}].$$

By a simple change of variables we see that

$$\int_{|z|<R/2} k^\alpha u_0(k^\beta z)dz = 2^{-s}\,|B_R(0)|^{s/N}\,|B_\rho(0)|^{-s/N}\int_{|z|<\rho} u_0(z)dz$$

where we have set $\rho = \frac{1}{2}Rk^\beta$. Thus for sufficiently large $\rho$ we have, using (GAV):

$$2^{-s}\,|B_R(0)|^{s/N}\frac{l}{2} \leqq c[R^{N+2/(m-1)} + [k^\alpha u(k^\beta x, k)]^{1+(m-1)N/2}]$$

and then rearranging the terms:

$$[k^\alpha u(k^\beta x, k)]^{1+(m-1)N/2} \geqq 2^{-s-1}c^{-1}\,|B_1(0)|^{s/N} l R^s - R^{N+2/(m-1)}. \tag{1.8}$$

Since $s \in [0, N + 2/(m-1))$ then the right hand side achieves a unique positive maximum at $\bar{R} = k_4' l^{(m-1)\beta}$, the value of the maximum is $k_5' l^{\beta[N(m-1)+2]}$ and $k_4'$ and $k_5'$ can be computed explicitly in terms of $m$, $N$, and $s$. Fixing the value of $R$ at $\bar{R}$ the condition of $\rho$ being sufficiently large requires that $k$ is sufficiently large. In this way we obtain from (1.8)

$$u(k^\beta x, k) \geqq k_5 l^{2\beta} k^{-\alpha}$$

which holds for $k$ sufficiently large, and $|x| < \bar{R}/2 = \frac{1}{2} k_4' l^{(m-1)\beta}$. Renaming $k$ as $t$ we arrive at the desired result.

Proof of Theorem B, part 1, Sufficiency of (H). We utilize again the family of similarity solutions of (PM), $w_k(x,t) = k^\alpha u(k^\beta x, kt)$, introduced in the proof of the lemma above. For some arbitrarily fixed positive $R$ and $T$ we set $Q = B_R(0) \times (0, T)$.

*Step I.* (Interior estimate) Recalling the definition of the norm $\|\cdot\|_{r,\mu}$ in the introduction, we have

$$\| w_k(\cdot, 0) \|_{R,\mu} = \| u_0 \|_{k^\beta R, \mu}$$

which holds for all $R > 0$, $k > 1$ and $\mu \in [0,1]$. Using the monotonicity of the norm $\|\cdot\|_{r,\mu}$ in $r$ we see that for $k > 1$,

$$\| w_k(\cdot, 0) \|_{R,\mu} \leqq \| u_0 \|_{R,\mu}.$$

Together with (1.2) then we have

$$\begin{aligned} \| w_k(\cdot, 0) \|_{R,1} &\leqq R^{-(1-\mu)[2/(m-1)+N]} \| w_k(\cdot, 0) \|_{R,\mu} \\ &\leqq R^{-(1-\mu)[2/(m-1)+N]} \| u_0 \|_{R,\mu}. \end{aligned}$$

Using this in (1.6) (applied to $w_k$) we obtain for $R > 1$

$$w(x,t) \leqq c_2 t^{-\lambda} R^{2\mu/(m-1)} \| u_0 \|_{R,\nu}^{2\lambda/N}$$

for all $|x| < R$ and

$$t < c_1^{(1-\mu)[2+N(m-1)]} \| u_0 \|_{R,\mu}^{1-m}.$$

We write this for short:

$$(1.9) \qquad w(x,t) \leqq k_6 t^{-\lambda}, \qquad |x| < R, \quad t < T.$$

Note that $k_6$, $R$, and $T$ are independent of $k$.

Next we derive a gradient estimate for $w_k$ using (1.42) in [5]:

$$\int_{|x|<R} |\nabla w_k(x,t)^{(p+m-1)/2}|^2 dx$$

$$\leqq cp\left[\frac{1}{t}\int_{|x|<2R} w_k(x,t)^p dx + \frac{1}{R^2}\int_{|x|<2R} w(x,t)^{p+m-1} dx\right].$$

Set $p = m+1$, proceed as in the proof of proposition 1.6 in [5] to obtain an estimate for $\int_{t_1}^{t_2}\int_{|x|<R}[(w_k^m)_t]^2 dxdt$ which is independent of $k$. These estimates provide sufficient compactness to conclude that

$$(1.10) \qquad w_{k_n}(x,t) \to q(x,t) \qquad \text{a.e. in } Q$$

for some sequence $k_n$ and for some function $q$, and $q$ satisfies (PM) in $Q$ in the sense of distributions.

*Step II.* (Boundary estimate) Here we will establish that

$$\lim_{t\to 0}\int_{\mathbf{R}^N} q(x,t)\varphi(x)dx = l\int_{\mathbf{R}^N} \Lambda_s(x)\varphi(x)dx$$

for all $\varphi \in C^\infty(\mathbf{R}^N)$ with support in $\{|x|<R\}$. From this and from the fact that $q$ satisfies (PM) in $\mathscr{D}'(Q)$ we will conclude via Theorem U in [5] the uniqueness of $q$ in the sense of independence on $R$ and the particular subsequence $k_n$ in (1.10). For this purpose multiply $w_{kt} = \Delta w_k^m$ by a function $\varphi$ in $C^\infty(Q)$ which vanishes near the boundary of $Q$ except on the side $t=0$ and obtain, after integrating by parts,

$$(1.11) \quad \int_0^T\int_{\mathbf{R}^N} w_k\frac{\partial\varphi}{\partial t}dxdt + \int_0^T\int_{\mathbf{R}^N} w_k^m\Delta\varphi dxdt + \int_{\mathbf{R}^N} w_k(x,0)\varphi(x,0)dx = 0.$$

The following lemma is used to pass to the limit $k\to\infty$ in (1.11).

LEMMA 1.6. *Let $Q=\{|x|<R\}\times(0,T)$ and $w_k(x,t)$ a family of measurable functions on $Q$ such that as $k\to\infty$*

(i) *$w_k \to q$ a.e. in $Q$,*

(ii) *$\int_{|x|<R} w_k(x,t)dx \leqq b_1$ for $t\in[0,t]$,*

(iii) *$w_k(x,t) \leqq b_2 t^{-\delta}$ for $t \in (0,t)$,*

*where $b_1$, $b_2$, $\delta$ are positive constants independent of $k$. Suppose that $(m-1)\delta<1$ for some $m>1$. Then*

$$(1.12) \qquad \lim_{k\to\infty}\int_0^T\int_{|x|<R} w_k^m\varphi dxdt = \int_0^T\int_{|x|<R} q^m\varphi dxdt$$

*for all test functions $\varphi$ as before.*

REMARK. Under the stronger hypothesis $m\delta < 1$, (1.12) follows immediately from (i), (iii), and the Lebesgue dominated convergence theorem.

PROOF OF THE LEMMA. By the Lebesgue dominated convergence theorem we have

$$\lim_{k\to\infty}\int_\varepsilon^T\int_{|x|<R} w_k^m\varphi dxdt = \int_\varepsilon^T\int_{|x|<R} q^m\varphi dxdt$$

for any $\varepsilon > 0$. Therefore it is sufficient for our purposes to show that

$$\lim_{\varepsilon\to 0}\int_0^\varepsilon\int_{|x|<R} w_k^m\varphi dxdt = 0 \tag{1.13}$$

*uniformly* in $k$. For this, choose $p > 1$ such that $(mp-1)\delta < 1$. Then

$$\begin{aligned}\int_Q w_k^{mp}\varphi dxdt &\leqq c\int_0^T\int_{|x|<R} w_k^{mp-1}w_k dxdt\\ &< cb_2^{mp-1}\int_0^T t^{-(mp-1)\delta}\int_{|x|<R} w_k dxdt\\ &\leqq cb_1b_2^{mp-1}\int_0^T t^{-(mp-1)\delta}dt\\ &\leqq c' < \infty, \qquad \text{independent of } k.\end{aligned}$$

Since $p > 1$ then (1.13) follows from the Hölder inequality.

To apply the lemma to our problem, let us recall that the assumptions (i) and (iii) have been established in (1.10) and (1.9), thus identify $\delta$ with $\lambda = N/[(m-1)N+2]$ which satisfies the requirement $(m-1)\lambda < 1$. To verify (ii), we first observe that

$$\begin{aligned}w_k(0,t+1) &= k^\alpha u(0,k(t+1))\\ &= (t+1)^{-\alpha}[(t+1)k]^\alpha u(0,k(t+1))\end{aligned}$$

which, by (1.4), is bounded independently of $k$ for all $t\in(0,T)$. Then by the fundamental estimate of Aronson and Caffarelli [3]

$$\int_{|x|<R} w_k(x,t)dx \leqq c[R^{N+2/(m-1)} + w_k(0,t+1)^{1+(m-1)N/2}]$$

which establishes hypothesis (ii) of Lemma 1.6. Applying the lemma we pass to the limit, as $k\to\infty$, the nonlinear term in (1.11).

A similar, but simpler argument justifies the limits of the remaining terms, thus we obtain from (1.11), using (H):

$$\int_0^T \int_{\mathbf{R}^N} q \frac{\partial \varphi}{\partial t} dxdt + \int_0^T \int_{\mathbf{R}^N} q^m \Delta \varphi dxdt + \int_{\mathbf{R}^N} l\Lambda_s(x)\varphi(x,0)dx = 0,$$

hence $q$ satisfies (PM) in the sense of distribution and takes on the value $l\Lambda_s(x)$ at $t = 0$.

*Step III.* (Conclusion) By the equicontinuity result of DiBenedetto [8] the estimate (iii) in Lemma 1.6 implies that the family $\{w_k(x,t)\}$ is relatively compact in the class of continuous functions as long as we stay away from $t = 0$. Therefore we can strengthen the convergence in (1.10) to conclude that

$$|w_k(x,t) - q(x,t)| \to 0 \quad \text{uniformly on } |x| < R \quad \text{as } k \to \infty,$$

$$\text{for all } t \text{ in } [\varepsilon, T], \qquad \varepsilon > 0.$$

So set $t = 1$, then $k = t$, to obtain

$$|t^\alpha u(t^\beta x, t) - q(x,1)| \to 0, \qquad |x| < R$$

as $t \to \infty$. Therefore

$$t^\alpha \, |u(y,t) - t^{-\alpha} q(t^{-\beta} y, 1)| \to 0, \qquad |y| < Rt^\beta.$$

Finally let's note that $t^{-\alpha} q(t^{-\beta} y, 1) = q(y,t)$ to finish the proof.

Proof of Theorem B, part 2, Necessity of (H). Here we consider a solution $u(x,t)$ of (PM) with a nonnegative initial condition in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$. Suppose that for some $s \in [0, 2/(m-1) + N)$, $R > 0$, $l > 0$, we have

$$\lim_{t \to \infty} \sup_{|x| < Rt^\beta} t^\alpha \, |u(x,t) - q(x,t)| = 0. \tag{1.14}$$

Then we will show that (H) holds. Recall that $q$, $\alpha$, and $\beta$ are defined in terms of $s$ and $l$ in the introduction of this paper.

*Step I.* Let $w_k(x,t) = k^\alpha u(k^\beta x, kt)$, $k > 1$, as before. For any $0 \leqq \tau < T$ let

$$Q_{\tau,T,R} = \{x \in \mathbf{R}^N : |x| < R\tau^\beta\} \times (\tau, T).$$

Here we show that for any $\tau > 0$, $T > \tau$, $w_k$ converges to $q$ uniformly on $Q_{\tau,T,R}$ as $k \to \infty$:

$$\lim_{k \to \infty} \sup_{(x,t) \in Q_{\tau,T,R}} |w_k(x,t) - q(x,t)| = 0. \tag{1.15}$$

To see this, rewrite (1.14) as

$$\sup_{|x| < Rt^\beta} t^\alpha \, |u(x,t) - q(x,t)| < \varepsilon \qquad \text{for } t > t(\varepsilon).$$

Therefore for any $k > 0$ we have

$$\sup_{|k^\beta x| < R(kt)^\beta} (kt^\alpha) | u(k^\beta x, kt) - q(k^\beta x, kt) | < \varepsilon$$

provided that $kt > t(\varepsilon)$. Now since $k^\alpha q(k^\beta x, kt) = q(x,t)$, we obtain, for all $t \in (\tau, T)$,

$$\sup_{|x| < R\tau^\beta} | w_k(x,t) - q(x,t) | < \frac{\varepsilon}{T^\alpha}$$

provided that $k > (1/\tau) t(\varepsilon)$, which proves (1.15).

*Step II.* For any positive numbers $\varepsilon$ and $r$, there exists a positive number $\tau(\varepsilon, r)$, independent of $k$, such that

$$\int_0^{\tau(\varepsilon,r)} \int_{|x|<r} w_k(x,t) dx dt < \varepsilon, \tag{1.16}$$

$$\int_0^{\tau} \int_{|x|<r} [w_k(x,t)]^m dx dt < \varepsilon, \tag{1.17}$$

for all $k \geqq 1$. We have already proved a close version of these statements in (1.13) so we omit the proof here. Essentially an identical argument also shows that (1.16) and (1.17) remain true with $w_k(x,t)$ replaced with $q(x,t)$.

*Step III.* Fix $\varepsilon > 0$ and take $\tau = \min\{1, \tau(\varepsilon, R)\}$. For an arbitrary $T > 1$ let $Q_0 = Q_{0,T,R}$, $Q_1 = Q_{0,\tau,R}$, $Q_2 = Q_{\tau,T,R}$. Let $\psi \in C^\infty(\mathbf{R}^N)$ be a function which vanishes in a neighborhood of the boundary of the cylinder $Q_0$, except on the base $t = 0$. Then since $w_k$ and $q$ are both weak solutions of (PM) we have

$$\begin{aligned} &\int_{Q_0} \frac{\partial \psi}{\partial t}(w_k - q) dx dt + \int_{Q_0} (\Delta\psi)(w_k^m - q^m) dx dt \\ &\quad + \int_{\mathbf{R}^N} \psi(x,0)[w_k(x,0) - q(x,0)] dx = 0. \end{aligned} \tag{1.18}$$

Recalling that $Q_0 = Q_1 \cup Q_2$, the first two integrals in (1.18) can be estimated by a multiple of $\varepsilon$ by choosing $k$ large and using the results of the previous two steps. Hence the last integral tends to zero as $k \to \infty$ for each choice of $\psi$. Let $\varphi(x) = \psi(x,0)$; recall the definition of $w_k$ and $q$ to obtain

$$\lim_{k\to\infty} \int_{\mathbf{R}^N} \varphi(x)[k^\alpha u_0(k^\beta x) - l\Lambda_s(x)] dx = 0.$$

Let $\xi = k^\beta$. Then $k^\alpha = \xi^{N-s}$, whence (H) follows.

REMARK. Note that we assumed *a priori* that $u_0$, the initial trace, is a locally integrable function. The trace in general is a measure and it has been studied in detail in the works of Aronson and Caffarelli [3] and Dahlberg and Kenig [7]. An appropriate version of Theorem B can be given in this general case. We do not give the details here.

## §2. A counterexample

Here we show by means of an example the inadequacy of the average condition (AV) to characterize the limiting state of the solutions of (PM). Specifically, consider the initial value problem

$$\begin{aligned} &u_t = (u^m)_{xx} \qquad\qquad m > 1, \\ &u(x,0) = \begin{cases} 0 & \text{when } x < 0, \\ 1 & \text{when } x > 0. \end{cases} \end{aligned} \tag{2.1}$$

The "average" of the initial data, measured in the sense of (AV), clearly exists and is equal to $\frac{1}{2}$. However the solution $u(x,t)$ of (2.1) does not approach $\frac{1}{2}$ as $t \to \infty$, contrary to what one may expect based on the experience with the heat equation.

THEOREM 2.1. *There exists a constant $\kappa_m$ such that*

$$u(x,t) \to \kappa_m \qquad \text{as } t \to \infty \tag{2.2}$$

*uniformly on compact $x$ intervals. Moreover $\kappa_m$ converges to* 1 *as $m \to +\infty$.*

REMARK 2.2. (i) The convergence in (2.2) is in fact stronger. It holds uniformly on sets $\{x \in \mathbf{R} : |x| < ct^{\sigma}\}$ for arbitrary $c > 0$ and $\sigma \in [0, \frac{1}{2})$.

(ii) Note that since $u_0(\xi x) \equiv u_0(x)$ for all $\xi > 0$ then the hypothesis (H) does not hold for this $u_0$.

PROOF. The equation and the initial data in (2.1) are invariant under the group of transformations $x \to \xi x$, $t \to \xi^2 t$, hence the solution has the form $u(x,t) = f(x/\sqrt{t})$. A simple calculation reveals that $f$ satisfies the ordinary differential equation

$$(f^m)'' + \tfrac{1}{2}\eta f' = 0, \tag{2.3}$$

$$f(-\infty) = 0, \qquad f(+\infty) = 1, \tag{2.4}$$

where the prime indicates $d/d\eta$.

*Initial* value problems for equation (2.3) have been studied extensively in [10] and [11] for $\eta > 0$. Here we use the invariance with respect to the reflection $\eta \to -\eta$ and some estimates of [11] and [12] to study the *boundary* value problem (2.3), (2.4). It follows from a combination of results of these two papers that there exists a continuous function $f$ and a constant $a > 0$ (depending on $m$) such that $f(\eta) = 0$ for $\eta \leqq -a$, $f(\eta)$ is monotone and increasing for $\eta > -a$, $f(\eta) \to 1$ as $\eta \to +\infty$, and $f$ solves (2.3) in the sense of distribution $\mathcal{D}'(\mathbf{R})$. In fact $f$ satisfies (2.3) in the classical sense on $\mathbf{R} - \{a\}$ and $(f^m)'(-a) = 0$, however $f'$ blows up at $-a$ when $m > 2$. Here we will need the following estimates, from [11], proposition 1, and [12], equation (13):

$$2^{1/m}\left(\frac{m-1}{4m}\right)^{1/(m-1)} a^{2/(m-1)} \leqq f(0), \tag{2.5}$$

$$f^{(m+1)/2}(\eta) \leqq f^{(m+1)/2}(0) + (4m)^{-1/2}(f^m)'(0), \qquad \eta > 0. \tag{2.6}$$

Now integrate (2.3) on $(-a, 0)$ to obtain

$$(f^m)'(0) = \tfrac{1}{2}\int_{-a}^{0} f(\eta)\,d\eta.$$

In view of the monotonicity of $f$ we then have $(f^m)'(0) \leqq \frac{1}{2}af(0)$. Using (2.5) we eliminate $a$:

$$(f^m)'(0) \leqq 2^{-(m-1)/2}\left(\frac{m}{m-1}\right)^{1/2} f(0)^{(m+1)/2},$$

then using this we eliminate $(f^m)'(0)$ in (2.6). We thus obtain for all $\eta > 0$:

$$f(\eta) \leqq [1 + 2^{-(3m-1)/2m}(m-1)^{-1/2}]^{2(m-1)} f(0).$$

Letting $\eta \to +\infty$ and recalling that $f(+\infty) = 1$, we arrive at

$$f(0) \geqq [1 + 2^{-(3m-1)/2m}(m-1)^{-1/2}]^{-2/(m-1)}. \tag{2.7}$$

By the monotonicity of $f$, $f(0) < 1$, hence letting $m \to \infty$ in (2.7) we see that $f(0) \to 1$.

Let $f(0) = \kappa_m$. Since $u(x,t) = f(x/\sqrt{t})$, then for $x$ in compact sets we have $u(x,t) \to \kappa_m$ uniformly in $x$, finishing the proof of the theorem.

Remark. Theorem 2.1 shows that the large time limit of $u(x,t)$ is greater than $\frac{1}{2}$ provided that $m$ is sufficiently large. Since this paper was submitted for publication, stronger versions of the theorem have emerged in connection with some related work. A comparison argument in Ghosh and Rostamian [10] readily shows that $\kappa_m > \frac{1}{2}$ if $m > 3$. A more delicate argument due to Bertsch and Peletier recorded in Alikakos [1] proves the optimal result $k_m > \frac{1}{2}$ for all $m > 1$.

**ppendix**

PROOF OF PROPOSITION C. (i) (H) $\Rightarrow$ (**). We take a smooth open set $\Omega \subset \mathbf{R}^N$ and we verify (**). In addition to the notation $\xi\Omega$ introduced in the statement of Proposition C, we also associate with $\Omega$ a family of nested open sets $\{\Omega_\delta : \delta \in \mathbf{R}\}$ with the following properties:

(a) $\Omega_0 = \Omega$,

(b) $\overline{\Omega_\delta} \subset \Omega_{\delta'}$ if $\delta < \delta'$,

(c) $|\Omega_\delta| / |\Omega| \to 1$ as $\delta \to 0$.

Here $|\Omega_\delta|$ denotes the measure of the set $\Omega_\delta$.

Fix a $\delta > 0$ and define $\varphi_{-\delta} \in C_0^\infty(\mathbf{R}^N)$ such that $0 \leqq \varphi_{-\delta} \leqq 1$, $\varphi_{-\delta}(x) = 1$ when $x \in \Omega_{-\delta}$, $\varphi_{-\delta}(x) = 0$ when $x \in \mathbf{R}^N - \Omega$. Now for any $\xi > 0$ we have

$$|\xi\Omega|^{-s/N} \int_{\xi\Omega} u_0(x)dx = |\Omega|^{-s/N} \int_\Omega \xi^{N-s} u_0(\xi x)dx$$

$$\geqq |\Omega|^{-s/N} \int_\Omega \xi^{N-s} u_0(\xi x)\varphi_{-\delta}(x)dx.$$

Let $\xi \to \infty$, use (H) to obtain

$$\liminf_{\xi\to\infty} |\xi\Omega|^{-s/N} \int_{\xi\Omega} u_0(x)dx \geqq |\Omega|^{-s/N} \int_{\Omega_{-\delta}} \Lambda_s(x)dx.$$

Essentially the same computation with $\varphi_\delta$ gives:

$$\limsup_{\xi\to\infty} |\Omega|^{-s/N} \int_{\xi\Omega} u_0(x)dx \leqq |\Omega|^{-s/N} \int_{\Omega_\delta} \Lambda_s(x)dx.$$

As $\delta \to 0$, the integrals on the right hand side tend to $\int_\Omega \Lambda_s(x)dx$, whence (**) follows.

(ii) (**) $\forall\Omega \Rightarrow$ (H). Here we assume that (**) holds for all $\Omega \subset \mathbf{R}^N$ and we verify (H). It is convenient to introduce the notation

$$R(\xi, \Omega, f) = |\xi\Omega|^{-s/N} \int_{\xi\Omega} f(x)dx$$

$$= |\Omega|^{-s/N} \int_\Omega \xi^{N-s} f(\xi x)dx.$$

Thus (**) reads

$$\lim R(\xi, \Omega, u_0) = lR(1, \Omega, \Lambda_s).$$

positive $\varepsilon$ and partition $\Omega$ into a finite, disjoint union of subsets $\Omega$ $\Omega = \bigcup_{i=1}^{n} \Omega^i$.

Take $x_i$ in $\Omega^i$, $i = 1, 2, \ldots, n$, in such a way that

(a) $$\left| \int_\Omega \Lambda_s(x)\varphi(x)dx - \sum_{i=1}^{n} \varphi(x_i) \int_{\Omega^i} \Lambda_s(x)dx \right| < \varepsilon.$$

(b) $$\sup_{x \in \Omega^i} |\varphi(x) - \varphi(x_i)| < \varepsilon, \quad \forall i.$$

(c) Choose $M(\varepsilon)$ such that (using (**))

$$|R(\xi, \Omega^i, u_0) - lR(1, \Omega^i, \Lambda_s)| < \varepsilon \qquad \text{if } \xi > M(\varepsilon), \quad \forall i.$$

Now we write down a chain of inequalities and finally combine them. From (a) we have

$$\left| \int_\Omega l\Lambda_s(x)\varphi(x)dx - \sum_{i=1}^{n} l\varphi(x_i)|\Omega^i|^{s/N} R(1, \Omega^i, \Lambda_s) \right| < l\varepsilon. \tag{A.1}$$

Now using (c), and taking $\xi > M(\varepsilon)$:

$$\begin{aligned} &\left| \sum_{i=1}^{n} l\varphi(x_i)|\Omega^i|^{s/N} R(1, \Omega^i, \Lambda_s) - \sum_{i=1}^{n} \varphi(x_i)|\Omega^i|^{s/N} R(\xi, \Omega^i, u_0) \right| \\ &< \varepsilon \sum_{i=1}^{n} |\Omega^i|^{s/N} \varphi(x_i). \end{aligned} \tag{A.2}$$

Next, recall the definition of $R$, and use (b) to get

$$\begin{aligned} &\left| \sum_{i=1}^{n} \varphi(x_i)|\Omega^i|^{s/N} R(\xi, \Omega^i, u_0) - \int_\Omega \xi^{N-s} u_0(\xi x)\varphi(x)dx \right| \\ &= \left| \sum_{i=1}^{n} \int_{\Omega^i} [\varphi(x_i) - \varphi(x)]\xi^{N-s} u_0(\xi x)dx \right| \\ &\leqq \varepsilon \int_\Omega \xi^{N-s} u_0(\xi x)dx = \varepsilon\, |\Omega|^{s/N} R(\varepsilon, \Omega, u_0). \end{aligned} \tag{A.3}$$

Combine (A.1), (A.2) and (A.3) to get

$$\begin{aligned} &\left| \int_\Omega l\Lambda_s(x)\varphi(x)dx - \int_\Omega \xi^{N-s} u_0(\xi x)\varphi(x)dx \right| \\ &\leqq \varepsilon l + \varepsilon \sum_{i=1}^{n} |\Omega^i|^{s-N} \varphi(x_i) + \varepsilon\, |\Omega|^{s/N} R(\xi, \Omega, u_0) \end{aligned}$$

provided that $\xi > M(\varepsilon)$. This finishes the proof.

PROOF OF PROPOSITION D. We assume that $u_0$ is radial, i.e.,

$$u_0(x) = f(r), \qquad r = (x_1^2 + \cdots + x_N^2)^{1/2}$$

for some function $f$. Using (GAV) we obtain (H). For this, fix any $c_2 > c_1 > 0$. A straightforward computation shows that

$$\begin{aligned}\xi^{N-s}\int_{c_1}^{c_2} f(\xi r)r^{N-1}dr &= \frac{c_2^2}{N\,|\,B_1(0)|^{1-s/N}}\,|\,B_{c_2\xi}(0)|^{-s/N}\int_{|x|<c_2} u_0(x)dx \\ &\quad - \frac{c_1^s}{N\,|\,B_1(0)|^{1-s/N}}\,B_{c_1\xi}(0)^{-s/N}\int_{|x|<c_1\xi} u_0(x)dx \\ &\to \frac{(c_2^s - c_1^s)l}{N\,|\,B_1(0)|^{1-s/N}} \qquad \text{as } \xi\to\infty\end{aligned} \tag{A.4}$$

by (GAV). Now take a test function $\varphi \in C_0(\mathbf{R}^N)$. Without loss of generality assume that $\varphi$ is supported in $B_1(0)$. Take an arbitrary positive integer $n$ and let $r_k = k/n$, $k = 1, 2, \ldots, n$. Introduce spherical coordinates $(r, \Theta)$ in $\mathbf{R}^N$ and compute

$$\begin{aligned} I &\equiv \xi^{N-s}\int_{\mathbf{R}^N} u_0(\xi x)\varphi(x)dx \\ &= \xi^{N-s}\sum_{k=1}^{n}\int_{r_{k-1}}^{r_k} r^{N-1}f(\xi r)\int_{S_1(0)} \varphi(r,\Theta)d\Theta dr.\end{aligned}$$

By the mean value theorem there are numbers $r_k^*$ in the interval $(r_{k-1}, r_k)$ such that

$$I = \xi^{N-s}\sum_{k=1}^{n}\left[\int_{S_1(0)} \varphi(r_k^*,\Theta)d\Theta\right]\int_{r_{k-1}}^{r_k} r^{N-1}f(\xi r)dr.$$

Using (A.4) we conclude that

$$\lim_{\xi\to\infty} I = \frac{l}{N\,|\,B_1(0)|^{1-s/N}}\sum_{k=1}^{n} r_k^{*N-1}\left[r_k^{*1-N}\int_{S_1(0)} \varphi(r_k^*,\Theta)d\Theta\right](r_k^s - r_{k-1}^s].$$

Passing to the limit as $r \to \infty$ and noting that $(r_k^s - r_{k-1}^s)/(r_k - r_{k-1}) - sr_k^{*s-1}$ tends to zero, we arrive at

$$\begin{aligned}\lim_{\xi\to\infty} I &= \frac{sl}{N\,|\,B_1(0)|^{1-s/N}}\int_0^1 r^{N-1}\left[r^{s-N}\int_{S_1(0)} \varphi(r,\Theta)d\Theta\right]dr \\ &= \int_{\mathbf{R}^N} \varphi(x)\Lambda_s(x)dx\end{aligned}$$

as required.

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